Return and risk

Lecture 1

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Section 1

Notation

R	return (a random variable)
r_i	realization of R if R is a discrete variable
r	realization of R if R is a continuous variable
$\mathrm{E}[R] = \mu_R$	expected value of R
$\mathrm{VAR}[R] = \sigma_R^2$	variance of R
σ_R	standard deviation of R

Section 2

Defining return

Defining return

Definition (Return)

Return is the growth rate of the investor's fortune.

Defining return

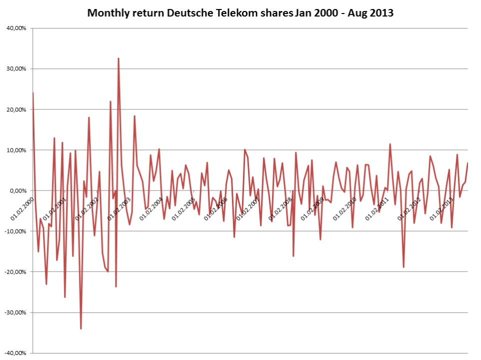
Time as discrete variable	Time as a continuous variable
$V_t = V_{t-1} \cdot (1 + r_t)$	$V(t) = V(t-1) \cdot \exp(r)$ $\exp(r) = \lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n$
$r_t = \frac{V_t}{V_{t-1}} - 1$	$r = \ln\left(V(t)\right) - \ln\left(V(t-1)\right)$

Section 3

Past return

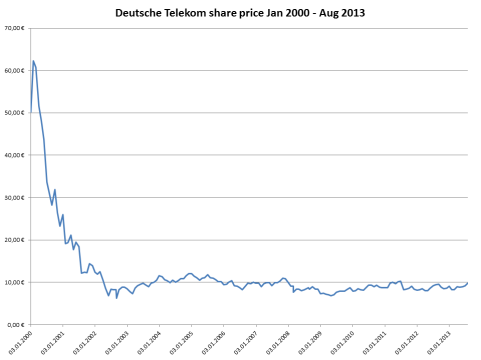
Past return

Development of returns and share prices over time



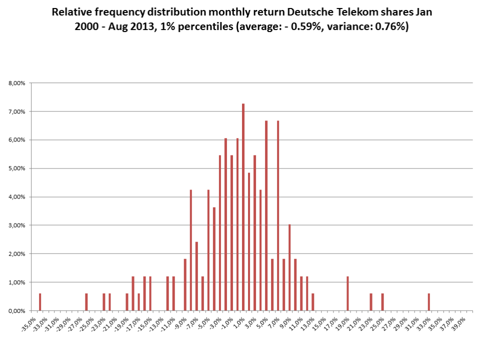
Past return

Development of returns and share prices over time



Past return

Frequency distribution



Past return

Measuring past performance

Measuring past performance

- Measure by geometric return
- Average return is misleading

Example

Sequence of returns: +50%, +30%, -100%

Past return

Measuring past performance

Definition (geometric average)

$$\left(\prod_{t=1}^{n}(1+r_t)\right)^{\frac{1}{n}} - 1 = \sqrt[n]{\prod_{t=1}^{n}(1+r_t)} - 1$$

In a limited liability economy return cannot be less than -100%.

Section 4

Future return

Future return

Randomness

Thesis

The future return R is a random variable.

Future return

Randomness

Randomness	Unpredictability
No objective reason for an event	Determinism. Cause and effect simply to complex to be understood.

Future return

Randomness

Definition (random variable R)

With return as a continuous variable:

$$\begin{aligned} R: \quad \Omega &\to \{r \in \mathbb{R} \mid r \geq -1\} \\ \omega &\mapsto R(\omega) \end{aligned}$$

With return as a discrete variable:

$$\begin{aligned} R: \quad \Omega &\to \{r_0, r_1, \ldots, r_n\} \\ \omega &\mapsto R(\omega) \end{aligned}$$

The sample space Ω is a set of outcomes ω.

Future return

Randomness

Examples for discrete sample spaces	
Random experiment	Sample space
'weather'	$\Omega = \{\text{sunshine, rain}\}$
'business cycle'	$\Omega = \{\text{expansion, boom, recession, depression}\}$
'stability of currency'	$\Omega = \{\text{currency persists, currency reform}\}$
'play dice'	$\Omega = \{1, 2, 3, 4, 5, 6\}$

Future return

Randomness

Example (Random variable)

$$R(\omega) = \begin{cases} 10\% & \text{if} \quad \omega = \text{`sunshine'} \\ -5\% & \text{if} \quad \omega = \text{`rain'} \end{cases}$$

Future return

Expected value

Definition (expected value of return)

- with discrete sample space:

$$\mathrm{E}[R] = \sum_{\omega \in \Omega} p(\omega) \cdot R(\omega)$$

Future return

Expected value

Definition (expected value of return)

- with return as a discrete variable:

$$E[R] = \sum_{i=0}^{n} p(R = r_i) \cdot r_i$$

- with return as a continuous variable (f is a probability density function):

$$E[R] = \int_{-\infty}^{+\infty} f(r) \cdot r \, dr$$

Future return

Variance

Thesis

The variance VAR of the return R is a measure of the risk attached to an individual asset.

Future return

Variance

Definition (variance)

- with return as a discrete variable:

$$\begin{aligned} VAR(R) &= E[R]\left[(R-\mu_R)^2\right] \\ &= \sum_{i=1}^{n} p(R=r_i)\cdot(\mu_R-r_i)^2 \end{aligned}$$

Remember notation: $E(R)=\mu_R$

Future return

Variance

Definition (variance)

- with continuous sample space (f is the probability density function):

$$\begin{aligned} \mathrm{VAR}[R] &= \mathrm{E}\left[(R-\mu_R)^2\right] \\ &= \int_{-\infty}^{+\infty} f(r)\cdot(r-\mu_R)^2\,dr \end{aligned}$$

Future return

Probability density

Definition (Probability density function)

P is a measure for probability. The return R is supposed to be a continuous random variable taking real numbers.

A function $f : \mathbb{R} \to [0, \infty[$ is called probability density function of R if

$$P(r_0 \leq R \leq r_1) = \int_{r0}^{r1} f(r)\,\mathrm{d}r$$

for all real numbers $r_0 < r_1$.

Future return

Probability density

In a limited liability economy, R cannot be less than -100%.

The domain of the probability density function of R is $[-1, \infty[$:

$$f : [-1, \infty[\to [0, \infty[$$

It is certain that the random return will take some value between -100% and infinity:

$$\int_{-1}^{\infty} f(r)\, \mathrm{d}r = 1$$

Future return

Probability density

Normal distribution of returns

Returns are often assumed to be normally distributed.

Advantage: The normal distribution $\mathcal{N}(\mu_R, \sigma_R^2)$ is described by just two parameters (namely μ_R and σ_R^2)).

Future return

Probability density

Normal distribution of returns

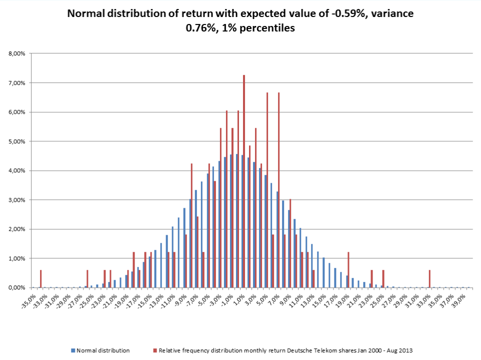
Problem:

- The domain of a normally distributed variable is $\mathbb{R}$ ($f : \mathbb{R} \to [0, \infty[$).
- But in a limited liability economy, R cannot be less than -100%, i.e. the domain of the probability density function of R should be $[-1, \infty[$.

Future return

Probability density

μ_R and σ_R^2 estimated on the basis of past frequency distributions.



Future return

Probability density

Log-normal distribution of share prices

Assumptions:

- The future return R is a continuous random variable.
- The current share price $V_0 = 1$ is a fixed, observed value.
- The future share price V_1 is a random variable.
- $V_1 = V_0 \cdot \exp(r)$ applies.
- R is normally distributed.

Conclusion:
V_1 is log-normally distributed.

Future return

Probability density