Leverage and diversification

Lecture 2

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r_0	riskless return
R_i	return (a random variable) of asset i with $i = \{0, \ldots, n\}$, $R_0 = r_0$
$\mathrm{E}[R_i] = \mu_i$	expected value of return of asset i
$\mathrm{COV}(R_i, R_j) = \sigma_{i,j}$	covariance of R_i and R_j
$\mathrm{VAR}(R_i) = \sigma_i^2 = \sigma_{i,i}$	variance of R_i

Section 2

Portfolio characteristics

Portfolio characteristics

Variance of return of single asset i

$$\text{VAR}(R_i) := \text{E}\left[(R_i - \mu_i)^2\right]$$

Covariance of return of two assets i and j

$$\begin{aligned} \text{COV}(R_i, R_j) &:= \text{E}\left[(R_i - \mu_i) \cdot (R_j - \mu_j)\right] \\ \Rightarrow \text{COV}(R_i, R_i) &= \text{VAR}(R_i) \end{aligned}$$

Portfolio characteristics

Random return of a portfolio

$$R_P = \sum_{i=0}^{n} x_i \cdot R_i$$

x_i is the share of the market value of asset i in the market value of the entire portfolio P:

$$\sum_{i=0}^{n} x_i = 1$$

Portfolio characteristics

Expected value of portfolio return

$$\mu_P = \sum_{i=0}^{n} x_i \cdot \mu_i = (x_0, x_1, \ldots x_n) \cdot \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix} = \vec{x}^T \cdot \vec{\mu}$$

Portfolio characteristics

Variance of portfolio return

$$\begin{aligned}
\mathrm{VAR}(R_P) &= \mathrm{E}\left[(R_P - \mu_P)^2\right] \\
&= \mathrm{E}\left[\left(\sum_{i=0}^{n} x_i \cdot R_i - \sum_{i=0}^{n} x_i \cdot \mu_i\right)^2\right] \\
&= \sum_{i=0}^{n} \sum_{j=0}^{n} x_i \cdot x_j \cdot \mathrm{COV}(R_i, R_j)
\end{aligned}$$

Portfolio characteristics

Variance of portfolio return

$$\begin{aligned}
\mathrm{VAR}(R_P) &= \sum_{i=0}^{n}\sum_{j=0}^{n} x_i \cdot x_j \cdot \mathrm{COV}(R_i, R_j) \\
&= (x_0, x_1, \ldots, x_n) \cdot \begin{pmatrix} \sigma_{0,0} & \sigma_{0,1} & \cdots & \sigma_{0,n} \\ \sigma_{1,0} & \sigma_{1,1} & \cdots & \sigma_{1,n} \\ \vdots & \vdots & & \vdots \\ \sigma_{n,0} & \sigma_{n,1} & \cdots & \sigma_{n,n} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} \\
&= \vec{x}^T \cdot C \cdot \vec{x}
\end{aligned}$$

Portfolio characteristics

Variance of portfolio return – portfolio with just two assets

$$\text{VAR}(R_P) = \sigma_P^2 = (x_1, x_2) \cdot \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Leftrightarrow \quad \sigma_P^2 = x_1^2\sigma_1^2 + 2 \cdot x_1 \cdot x_2 \cdot \sigma_{12} + x_2^2\sigma_2^2$$

Portfolio characteristics

Definition (correlation coefficient of returns)

$$\rho_{i,j} = \frac{\mathrm{COV}(R_i, R_j)}{\sigma_i \cdot \sigma_j}$$

Portfolio characteristics

The correlation coefficient $\rho_{i,j}$ falls between $[-1;1]$	
$\rho_{i,j} = 0$	no correlation
$\rho_{i,j} = 1$	perfect positive correlation
$\rho_{i,j} = -1$	perfect negative correlation

Section 3

Leverage & short selling

Leverage & short selling

Definitions

Definition (Leverage)

Leverage is an investment strategy.

- Aim of leverage: Increase of the expected value of returns.
- How to achieve leverage: Borrow money at the riskless rate and invest the borrowed amount in risky assets.

Leverage & short selling

Definitions

Definition (Short selling)

Selling assets that are not currently owned.

The usual intention is to purchase ('cover') the assets later at a lower price.

Leverage & short selling

Mathematics

Return of leveraged portfolio

$$\mu_P = x_0 \cdot r_0 + (1 - x_0) \cdot \mu_1$$

Leverage & short selling

Mathematics

Risk of leveraged portfolio

$$
\begin{aligned}
\sigma_P^2 = & \quad (x_0, 1 - x_0) \cdot \begin{pmatrix} 0 & 0 \\ 0 & \sigma_1^2 \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ 1 - x_0 \end{pmatrix} \\
\Leftrightarrow \quad \sigma_P = & \quad (1 - x_0) \cdot \sigma_1
\end{aligned}
$$

Leverage & short selling

Mathematics

Achievable μ_P-σ_P-combinations

$$\mu_P = r_0 + \frac{\mu_1 - r_0}{\sigma_1} \cdot \sigma_P$$

Leverage & short selling

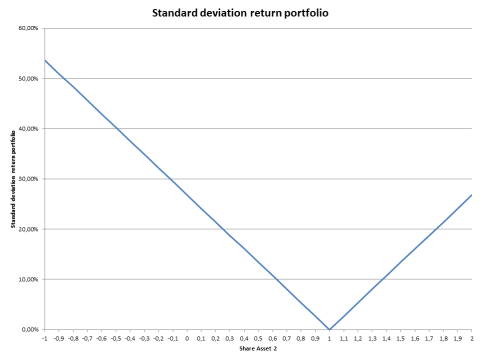
Numeric example

Numeric example

Probability	0,25	0,25	0,25	0,25
State of the environment	1	2	3	4
Asset 1	10,00%	20,00%	40,00%	80,00%
Asset 2	15,00%	15,00%	15,00%	15,00%

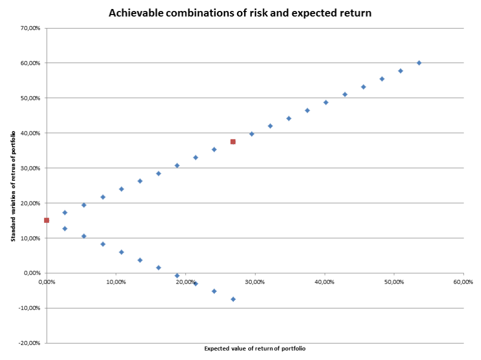
Leverage & short selling

Numeric example



Leverage & short selling

Numeric example



Leverage & short selling

Lessons

To remember

1. There is no ‘optimal portfolio’.
2. There is a trade-off between risk and return: You can only trade risk for return and vice versa.
3. Is 20% expected return really better than 10% expected return? – That depends on the risk!

Section 4

Diversification

Diversification

Definitions

Definition (Diversification)

Diversification is an investment strategy.

- Aim of diversification: Reduction of risk. The variance of the return on a portfolio of risky assets shall be lower than the sum of the variance of the returns on individual assets.
- How to achieve diversification: Combining risky assets. Correlation of the returns on these assets is less than $+1$.

Diversification

Definitions

Definition (Hedging)

Combination of risky assets. Correlation of the returns on these assets is less than 0 ($\rho_{i,j} < 0$).

Definition (Perfect hedge)

Creation of a zero risk portfolio from risky assets. Correlation of returns is -1 ($\rho_{i,j} = -1$).

Diversification

Mathematics – two risky assets

- Assume you got two risky assets earning an expected return of $\mu_1 = E(R_1)$ respectively $\mu_1 = E(R_1)$.
- Put a share of x_1 in the riskless asset and a share of $x_2 = 1 - x_1$ in the risky asset.
- Allow short selling of the risky asset ($x_1 < 0$ and $x_2 < 0$ is possible).

Diversification

Mathematics – two risky assets

Extreme scenario: Perfect positive correlation of risks

$$\rho_{i,j} = \frac{\mathrm{COV}(R_i, R_j)}{\sigma_i \cdot \sigma_j} = 1$$

$$\Leftrightarrow \quad \mathrm{COV}(R_i, R_j) = \sigma_i \cdot \sigma_j$$

Diversification

Mathematics – two risky assets

Extreme scenario: Perfect positive correlation of risks

Portfolio risk:

$$\sigma_P^2 = (x_1, x_2) \cdot \begin{pmatrix} \sigma_1^2 & \sigma_1 \cdot \sigma_2 \\ \sigma_1 \cdot \sigma_2 & \sigma_2^2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Leftrightarrow \quad \sigma_P^2 = (x_1 \cdot \sigma_1)^2 + 2 \cdot (x_1 \cdot \sigma_1) \cdot (x_2 \cdot \sigma_2) + (x_2 \cdot \sigma_2)^2$$

$$\Rightarrow \quad \sigma_P = x_1 \cdot \sigma_1 + x_2 \cdot \sigma_2$$

Diversification

Mathematics – two risky assets

Extreme scenario: Perfect positive correlation of risks

$$\rho_{i,j} = \frac{\mathrm{COV}(R_i, R_j)}{\sigma_i \cdot \sigma_j} = -1$$

$$\Leftrightarrow \quad \mathrm{COV}(R_i, R_j) = -\sigma_i \cdot \sigma_j$$

Diversification

Mathematics – two risky assets

Extreme scenario: Perfect negative correlation of risks

Portfolio risk:

$$\sigma_P^2 = (x_1, x_2) \cdot \begin{pmatrix} \sigma_1^2 & -\sigma_1 \cdot \sigma_2 \\ -\sigma_1 \cdot \sigma_2 & \sigma_2^2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Leftrightarrow \quad \sigma_P^2 = (x_1 \cdot \sigma_1)^2 - 2 \cdot (x_1 \cdot \sigma_1) \cdot (x_2 \cdot \sigma_2) + (x_2 \cdot \sigma_2)^2$$

$$\Rightarrow \quad \sigma_P = x_1 \cdot \sigma_1 - x_2 \cdot \sigma_2$$

Diversification

Mathematics – two risky assets

Extreme scenario: Perfect negative correlation of risks

The perfectly hedged portfolio ('synthetic' risk-free security) fulfils the condition:

$$\sigma_P = 0$$

$$\Rightarrow \quad x_1 \cdot \sigma_1 - x_2 \cdot \sigma_2 = 0$$

with

$$x_1 = 1 - x_2$$

Diversification

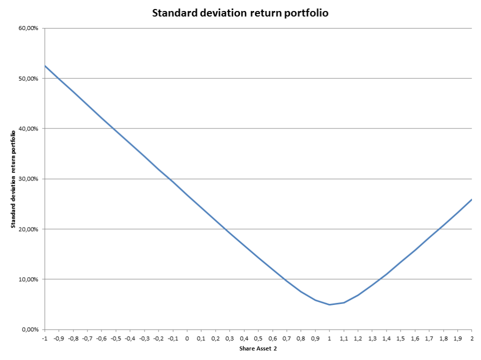
Numeric example – correlation 0.6

Numeric example – correlation 0.6

Probability	0,25	0,25	0,25	0,25
State of the environment	1	2	3	4
Asset 1	10,00%	20,00%	40,00%	80,00%
Asset 2	40,00%	30,00%	30,00%	40,00%

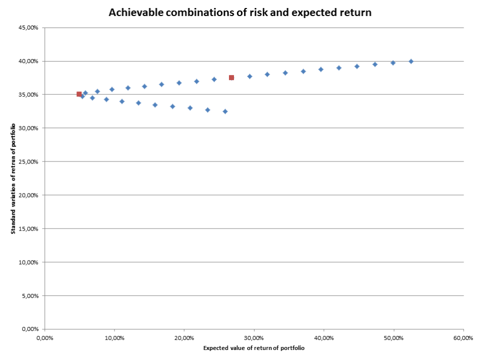
Diversification

Numeric example – correlation 0.6



Diversification

Numeric example – correlation 0.6



Diversification

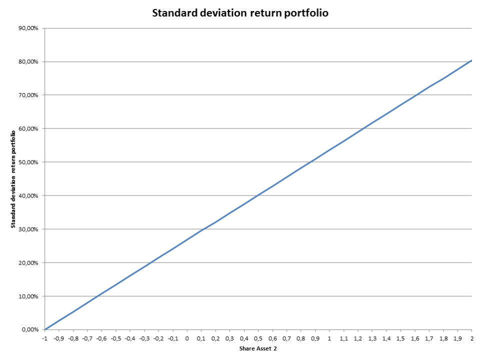
Numeric example – correlation 1.0

Numeric example – correlation 1.0

Probability	0,25	0,25	0,25	0,25
State of the environment	1	2	3	4
Asset 1	10,00%	20,00%	40,00%	80,00%
Asset 2	20,00%	40,00%	80,00%	160,00%

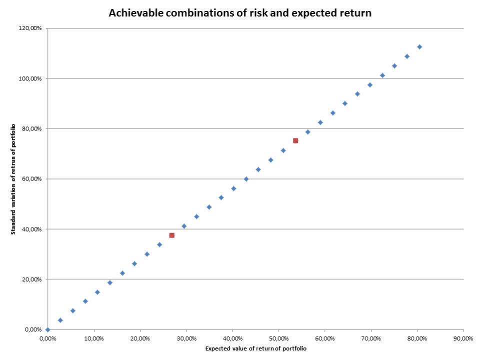
Diversification

Numeric example – correlation 1.0



Diversification

Numeric example – correlation 1.0



Diversification

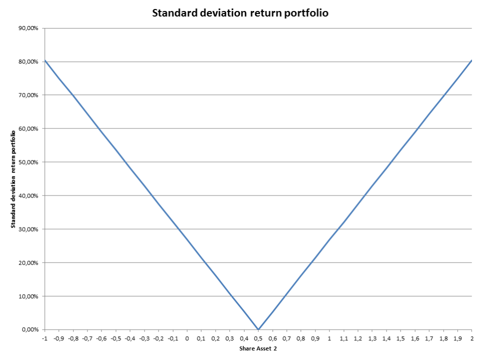
Numeric example – correlation -1.0

Numeric example – correlation -1.0

Probability	0,25	0,25	0,25	0,25
State of the environment	1	2	3	4
Asset 1	10,00%	20,00%	40,00%	80,00%
Asset 2	90,00%	80,00%	60,00%	20,00%

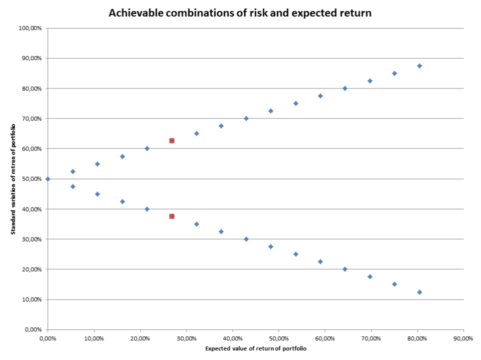
Diversification

Numeric example – correlation -1.0



Diversification

Numeric example – correlation -1.0



Diversification

Lessons

To remember

1. Don't separate decisions to invest in assets.
   - Consider correlation of risks.
2. Efficient portfolios can comprise assets that are dominated by other assets.
3. The risk of a diversified portfolio might be lower than the least risk in the portfolio.