Single Factor Models

Lecture 6

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Section 1

Assumptions

Assumptions

Assumption on the return R_i of every asset i

$$R_i = a_i + b_i \cdot F + \varepsilon_i$$

with

- F factor determining return that is common to multiple assets
- ε_i idiosyncratic component of return
- a_i component of return which is independent from F
- b_i factor sensitivity

Assumptions

Further assumptions

- $\mathsf{E}[\varepsilon_i] = 0$
- constant variance $\mathsf{VAR}(\varepsilon_{it}) = \mathsf{VAR}(\varepsilon_i)$
- $\mathsf{COV}(\varepsilon_i, F) = 0$
- $\mathsf{COV}(\varepsilon_i, \varepsilon_j) = 0 \quad \forall i \neq j$

Assumptions

Interpretation of shocks

Deviations from the expected return on asset i ($R_i \neq \mathrm{E}[R_i]$) are explained by:

1. a common shock, that is an unexpected movement in the common factor ($F \neq \mathrm{E}[F]$)
2. an idiosyncratic (or asset-specific) shock ($\varepsilon_i \neq 0$)

Assumptions

Interpretation of $\text{COV}(\varepsilon_i, F) = 0$

- idiosyncratic shock is uncorrelated with common shock
- otherwise common shock would explain part of idiosyncratic shock
- one could not call it idiosyncratic anymore

Assumptions

Interpretation of $\mathrm{COV}(\varepsilon_i, \varepsilon_j) = 0$

$\mathrm{COV}(\varepsilon_i, \varepsilon_j) = 0$ is what constitutes idiosyncratic risk.

Section 2

Return of distinct assets

Return of distinct assets

Expected return of asset

$$\mu_i = \mathrm{E}[R_i] = \mathrm{E}[a_i + b_i \cdot F + \varepsilon_i] = a_i + b_i \cdot \mathrm{E}[F]$$

Return of portfolio

Linear transformation of covariance / variance – Background

$$
\begin{aligned}
& \mathrm{COV}(R_i, R_j) \\
= \ & \mathrm{COV}(a_i + b_i \cdot F + \varepsilon_i, a_j + b_j \cdot F + \varepsilon_j) \\
= \ & \mathrm{E}\left[(a_i + b_i \cdot F + \varepsilon_i - \mathrm{E}[a_i + b_i \cdot F]) \cdot (a_j + b_j \cdot F + \varepsilon_j - \mathrm{E}[a_j + b_j \cdot F])\right] \\
= \ & \mathrm{E}\left[(b_i \cdot F + \varepsilon_i - \mathrm{E}[b_i \cdot F]) \cdot (b_j \cdot F + \varepsilon_j - \mathrm{E}[b_j \cdot F])\right] \\
= \ & b_i \cdot b_j \cdot \mathrm{E}\left[(F - \mathrm{E}[F])^2\right] + \mathrm{E}\left[(\varepsilon_i - \mathrm{E}[\varepsilon_i]) \cdot (\varepsilon_j - \mathrm{E}\left[\varepsilon_j\right])\right] \\
= \ & b_i \cdot b_j \cdot \mathrm{VAR}(F) + \mathrm{COV}(\varepsilon_i, \varepsilon_j)
\end{aligned}
$$

Note:

$$E\left[\varepsilon_j\right] = 0$$

Return of portfolio

Risk of distinct asset

$\forall i = j :$

$$\begin{cases} COV(R_i, R_j) & = & b_i \cdot b_j \cdot VAR(F) + COV(\varepsilon_i, \varepsilon_j) \\ VAR(R_i) & = & COV(R_i, R_i) \\ VAR(\varepsilon_i) & = & COV(\varepsilon_i, \varepsilon_i) \end{cases}$$

$$\Rightarrow \quad VAR(R_i) = b_i^2 \cdot VAR(F) + VAR(\varepsilon_i)$$

Return of distinct assets

Risk of distinct asset		
	$b_i^2 \cdot \mathrm{VAR}(F)$	systematic risk (synonym: factor risk)
$+$	$\mathrm{VAR}(\varepsilon_i)$	idiosyncratic (asset-specific) risk
$=$	$\mathrm{VAR}(R_i)$	total risk

Return of portfolio

Covariance between asset returns

$$\forall i \neq j :$$

$$\begin{cases} \mathrm{COV}(R_i, R_j) & = & b_i \cdot b_j \cdot \mathrm{VAR}(F) + \mathrm{COV}(\varepsilon_i, \varepsilon_j) \\ \mathrm{COV}(\varepsilon_i, \varepsilon_j) & = & 0 \end{cases}$$

$$\Rightarrow \quad \mathrm{COV}(R_i, R_j) = b_i \cdot b_j \cdot \mathrm{VAR}(F)$$

Return of portfolio

Covariance between asset returns

Covariances are explained entirely by the variance of F and the assets' different sensitivities to F:

$$\mathrm{COV}(R_i, R_j) = b_i \cdot b_j \cdot \mathrm{VAR}(F)$$

Conclusion: A SFM requires less data than Markowitz' portfolio selection theory.

Section 3

Return of portfolio

Return of portfolio

Portfolio return

$$R_P = a_P + b_P \cdot F + \varepsilon_P$$

with

$$\begin{aligned} a_P &= \sum_{i=1}^{n} x_i \cdot a_i \\ b_P &= \sum_{i=1}^{n} x_i \cdot b_i \\ \varepsilon_P &= \sum_{i=1}^{n} x_i \cdot \varepsilon_i \end{aligned}$$

Return of portfolio

Expected value of portfolio return

$$\mathrm{E}[R_P] = a_P + b_P \cdot \mathrm{E}[F]$$

Return of portfolio

Variance of return of a portfolio

$$
\begin{aligned}
&\mathrm{VAR}[R_P] \\
= \quad &(x_1, x_2, \ldots, x_n) \cdot \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} & \ldots & \sigma_{1,n} \\ \sigma_{2,1} & \sigma_{2,2} & \ldots & \sigma_{2,n} \\ \vdots & \vdots & & \vdots \\ \sigma_{n,1} & \sigma_{n,2} & \ldots & \sigma_{n,n} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
\end{aligned}
$$

Return of portfolio

Variance of return of a portfolio

Insert:

$$\begin{aligned} & \sigma_{i,i} && = \mathrm{VAR}(R_i) && = b_i^2 \cdot \mathrm{VAR}(F) + \mathrm{VAR}(\varepsilon_i) \\ \forall i \neq j: \quad & \sigma_{i,j} && = \mathrm{COV}(R_i, R_j) && = b_i \cdot b_j \cdot \mathrm{VAR}(F) \end{aligned}$$

Return of portfolio

Variance of portfolio return

$$\text{VAR}(R_P) = b_P^2 \cdot \text{VAR}(F) + \text{VAR}(\varepsilon_p)$$

with

$$\begin{aligned} b_P &= \sum_{i=1}^{n} x_i \cdot b_i \\ \text{VAR}(\varepsilon_P) &= \sum_{i=1}^{n} x_i^2 \cdot \text{VAR}(\varepsilon_i) \end{aligned}$$

Return of portfolio

Variance of portfolio return

Assumption: $x_1 = \ldots = x_n = \frac{1}{n}$

Conclusion: $\lim_{n \to \infty} (\text{VAR}(\varepsilon_P)) = b_P^2 \cdot \text{VAR}(F)$

Interpretation: Sound diversification eliminates specific risk.

Return of portfolio

Covariance between asset return and factor F

b_i is the sensitivity of R_i with regard to F:

$$
\begin{aligned}
\mathrm{COV}(R_i, F) &= \mathrm{COV}(a_i + b_i \cdot F + \varepsilon_i, F) \\
&= b_i \cdot \mathrm{VAR}(F) \\
\Leftrightarrow \quad b_i &= \frac{\mathrm{COV}(R_i, F)}{\mathrm{VAR}(F)}
\end{aligned}
$$

Section 4

SIM as a special SFM

SIM as a special SFM

Model outline

SIM – Basic assumption

$$ER_i = a_i + b_i \cdot ER_M + \varepsilon_i$$

with

$ER_i = R_i - r_0$	random excess return on asset i
$ER_M = R_M - r_0$	random excess return on indexed portfolio

SIM as a special SFM

Model outline

SIM – Basic assumption

Interpretation: The common component of unanticipated movements in individual assets is explained by unanticipated movements in the return on a broad index of securities, such as the S&P500 index.

In formal language:

$$F = R_M - r_0$$

SIM as a special SFM

Model outline

Risk premium of distinct asset

$$E[ER_i] = a_i + b_i \cdot E[ER_M]$$

with

$E[ER_i] = E[R_i] - r_0$ risk premium on asset i

$E[ER_M] = E[R_M] - r_0$ risk premium on indexed portfolio

SIM as a special SFM

Model outline

Risk of distinct asset

$$VAR(R_i) = b_i^2 \cdot VAR(R_M) + VAR(\varepsilon_i)$$

whereby the variance of returns equals the variance of excess returns:

$$\begin{aligned} VAR(R_i) &= VAR(ER_i) \\ VAR(R_M) &= VAR(ER_M) \end{aligned}$$

SIM as a special SFM

Model outline

Risk of distinct asset		
	$b_i^2 \cdot \text{VAR}(R_M)$	systematic risk (synonym: factor risk)
$+$	$\text{VAR}(\varepsilon_i)$	idiosyncratic (asset-specific) risk
$=$	$\text{VAR}(R_i)$	total risk

SIM as a special SFM

Comparison with CAPM

Estimates needed for...	
Markowitz' portfolio selection	Single Index Model
• riskless interest rate • n expected values of returns • n variances of returns • $(n^2 - n)/2$ covariances	• riskless interest rate • expected value of index • variance of index • a_i of all n assets • b_i of all n assets • variances of noise of all n assets
Total $(n^2 + 3 \cdot n + 2)/2$	Total $3 \cdot n + 3$

SIM as a special SFM

Comparison with CAPM

Compatibility of CAPM and SIM: Factor-sensitivity b_i

$$
\begin{aligned}
b_i &= \frac{\text{COV}(RP_i, RP_M)}{\text{VAR}(RP_M)} \\
\Leftrightarrow \quad b_i &= \frac{\text{COV}(R_i, R_M)}{\text{VAR}(R_M)}
\end{aligned}
$$

SIM as a special SFM

Comparison with CAPM

Compatibility of CAPM and SIM: Constant a_i

Risk premium of distinct asset according to SIM:

$$\begin{aligned} & \mathrm{E}[ER_i] = a_i + b_i \cdot \mathrm{E}[ER_M] \qquad |a_i = 0 \\ \Rightarrow \quad & \mathrm{E}[R_i] = r_0 + b_i \cdot (\mathrm{E}[R_M] - r_0) \end{aligned}$$

Conclusion: If $a_i = r_0$, the CAPM and the SFM yield the same result.

SIM as a special SFM

Comparison with CAPM

SIM – Interpretation of $a_i = 0$

Interpretation 1: There are no factors outside the model (i.e. factors other than $\mathrm{E}[ER_M]$) determining $\mathrm{E}[ER_i]$.

Interpretation 2: There are no opportunities for arbitrage, which is a weaker condition than the market equilibrium assumed by the CAPM (see next lecture on APT).